On defensive alliances and line graphs

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Abstract

Let Γ be a simple graph of size m and degree sequence $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$. Let $\mathcal{L}(\Gamma)$ denotes the line graph of Γ. The aim of this paper is to study mathematical properties of the alliance number, $a(\mathcal{L}(\Gamma)$, and the global alliance number, $\gamma_a(\mathcal{L}(\Gamma))$, of the line graph of a simple graph. We show that $\left\lceil \frac{\delta_n+\delta_{n-1}-1}{2} \right\rceil \leq a(\mathcal{L}(\Gamma)) \leq \delta_1$. In particular, if Γ is a δ-regular graph ($\delta > 0$), then $a(\mathcal{L}(\Gamma)) = \delta$, and if Γ is a (δ_1, δ_2)-semiregular bipartite graph, then $a(\mathcal{L}(\Gamma)) = \left\lceil \frac{\delta_1+\delta_2-1}{2} \right\rceil$. As a consequence of the study we compare $a(\mathcal{L}(\Gamma))$ and $a(\Gamma)$, and we characterize the graphs having $a(\mathcal{L}(\Gamma)) < 4$. Moreover, we show that the global-connected alliance number of $\mathcal{L}(\Gamma)$ is bounded by $\gamma_{ca}(\mathcal{L}(\Gamma)) \geq \left\lceil \sqrt{D(\Gamma)+m-1} - 1 \right\rceil$, where $D(\Gamma)$ denotes the diameter of Γ, and we show that the global alliance number of $\mathcal{L}(\Gamma)$ is bounded by $\gamma_a(\mathcal{L}(\Gamma)) \geq \left\lceil \frac{2m}{\delta_1+\delta_2+1} \right\rceil$. The case of strong alliances is studied by analogy.

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1 Introduction

The study of defensive alliances in graphs, together with a variety of other kinds of alliances, was introduced in [3]. In the referred paper was initiated the study of the mathematical properties of alliances. In particular, several bounds on the defensive alliance number were given. The particular case of global (strong) defensive alliance was investigated in[2] where several bounds on the global (strong) defensive alliance number were obtained.

In [4] were obtained several tight bounds on different types of alliance numbers of a graph, namely (global) defensive alliance number, (global) offensive alliance number and (global) dual alliance number. In particular, was investigated the relationship between the alliance numbers of a graph and its algebraic connectivity, its spectral radius, and its Laplacian spectral radius. A particular study of the alliance numbers, for the case of planar graphs, can be found in [5]. Moreover, for the study of offensive alliances we cite [1, 6].

The aim of this paper is to study mathematical properties of the alliance number and the global alliance number of the line graph of a simple graph. We begin by stating some notation and terminology. In this paper $\Gamma = (V, E)$ denotes a simple graph of order n and size m. The degree sequence of Γ will be denoted by $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$. Moreover, the degree of a vertex $v \in V$ will be denoted by $\delta(v)$. The line graph of Γ will be denoted by $\mathcal{L}(\Gamma) = (V_l, E_l)$. The degree of the vertex $e = \{u, v\} \in V_l$ is $\delta(e) = \delta(u) + \delta(v) - 2$. The subgraph induced by a set $S \subset V$ will be denoted by $\langle S \rangle$.

For a non-empty subset $S \subseteq V$, and any vertex $v \in V$, we denote by $N_S(v)$ the set of neighbors v has in S: $N_S(v) := \{u \in S : u \sim v\}$, Similarly, we denote by $N_{V \setminus S}(v)$ the set of neighbors v has in $V \setminus S$: $N_{V \setminus S}(v) := \{u \in V \setminus S : u \sim v\}$.

A nonempty set of vertices $S \subseteq V$ is called a *defensive alliance* if for every $v \in S$, $|N_S(v)| + 1 \geq |N_{V \setminus S}(v)|$. Equivalently, S is a defensive alliance if for every $v \in S$, $2|N_S(v)| + 1 \geq \delta(v)$. In this case, by strength of numbers, every vertex in S is *defended* from possible attack by vertices in $V \setminus S$. A defensive alliance S is called *strong* if for every $v \in S$, $|N_S(v)| \geq |N_{V \setminus S}(v)|$. Equivalently, S is a defensive alliance if for every $v \in S$, $2|N_S(v)| \geq \delta(v)$. In this case every vertex in S is *strongly defended*.

The *defensive alliance number* $a(\Gamma)$ (respectively, *strong defensive alliance number* $\hat{a}(\Gamma)$) is the minimum cardinality of any defensive alliance (respectively, strong defensive alliance) in Γ. A defensive alliance, S, in Γ is *minimal*

if no proper subset of S is a defensive alliance. A *minimum* defensive alliance is a minimal defensive alliance of smallest cardinality, i.e., $|S| = a(\Gamma)$.

A particular case of alliance, called global defensive alliance, was studied in [2]. A defensive alliance S is called *global* if it affects every vertex in $V \setminus S$, that is, every vertex in $V \setminus S$ is adjacent to at least one member of the alliance S. Note that, in this case, S is a dominating set. The *global defensive alliance number* $\gamma_a(\Gamma)$ (respectively, *global strong defensive alliance number* $\gamma_{\hat{a}}(\Gamma)$) is the minimum cardinality of any global defensive alliance (respectively, global strong defensive alliance) in Γ. Singular interest displays the global defensive alliances whose induced subgraph is connected. We define the *global-connected defensive alliance number*, $\gamma_{ca}(\Gamma)$, (respectively, *global-connected strong defensive alliance number* $\gamma_{c\hat{a}}(\Gamma)$) as the minimum cardinality of any global defensive alliance (respectively, global strong defensive alliance) in Γ whose induced subgraph is connected.

In this paper we show that the alliance number of $\mathcal{L}(\Gamma)$ is bounded by $\left\lceil \frac{\delta_n+\delta_{n-1}-1}{2} \right\rceil \le a(\mathcal{L}(\Gamma)) \le \delta_1$. In particular, if Γ is a δ-regular graph ($\delta > 0$), then $a(\mathcal{L}(\Gamma)) = \delta$, and if Γ is a (δ_1, δ_2)-semiregular bipartite graph, then $a(\mathcal{L}(\Gamma)) = \left\lceil \frac{\delta_1+\delta_2-1}{2} \right\rceil$. As a consequence of the study we compare $a(\mathcal{L}(\Gamma))$ and $a(\Gamma)$, and we characterize the graphs having $a(\mathcal{L}(\Gamma)) < 4$. In the case of global alliances, we show that the global alliance number of $\mathcal{L}(\Gamma)$ is bounded by $\gamma_a(\mathcal{L}(\Gamma)) \ge \left\lceil \frac{2m}{\delta_1+\delta_2+1} \right\rceil$ and the global-connected alliance number of $\mathcal{L}(\Gamma)$ is bounded by $\gamma_{ca}\left(\mathcal{L}(\Gamma)\right) \ge \left\lceil \sqrt{D(\Gamma)+m-1} - 1 \right\rceil$, where $D(\Gamma)$ denotes the diameter of Γ. In addition, the case of strong alliances is studied by analogy.

2 Defensive alliances and line graphs

Theorem 1. *Let Γ be a graph whose degree sequence is $\delta_1 \ge \delta_2 \ge \cdots \ge \delta_n$. Then $\left\lceil \frac{\delta_n+\delta_{n-1}}{2} \right\rceil \le \hat{a}(\mathcal{L}(\Gamma)) \le \delta_1$ and $\left\lceil \frac{\delta_n+\delta_{n-1}-1}{2} \right\rceil \le a(\mathcal{L}(\Gamma)) \le \delta_1$. Moreover, if Γ has a unique vertex of maximum degree, then $a(\mathcal{L}(\Gamma)) \le \delta_1 - 1$.*

Proof. If S_l denotes a strong defensive alliance in $\mathcal{L}(\Gamma)$, then $\forall e \in S$,

$$2(|S_l| - 1) \ge 2|N_{S_l}(e)| \ge \delta(e) \ge \delta_n + \delta_{n-1} - 2.$$

Therefore, the lower bound of $\hat{a}(\mathcal{L}(\Gamma))$ follows.

Let v be a vertex of maximum degree in Γ and let $S_v = \{e \in E : v \in e\}$. Thus, $\langle S_v \rangle \cong K_{\delta_1}$ and, as a consequence, $\forall e \in S_v$, $|N_{S_v}(e)| = \delta_1 - 1 \ge$

$\delta_2 - 1 \geq |N_{V_l \setminus S_v}(e)|$. Hence, $S_v \subset V_l$ is a strong defensive alliance in $\mathcal{L}(\Gamma)$. So, $\hat{a}(\mathcal{L}(\Gamma)) \leq \delta_1$.

The lower bound of $a(\mathcal{L}(\Gamma))$ is obtained by analogy to the previous case. Moreover, $a(\mathcal{L}(\Gamma)) \leq \hat{a}(\mathcal{L}(\Gamma)) \leq \delta_1$.

Suppose that $v \in V$ is the unique vertex of maximum degree in Γ. As above, let $S_v = \{e \in E : v \in e\}$. Let $e' \in S_v$ and let $S'_v = S_v \setminus \{e'\}$. Thus, $\langle S'_v \rangle \cong K_{\delta_1 - 1}$. Hence, $\forall e \in S'_v$, $|N_{S'_v}(e)| + 1 = \delta_1 - 1 \geq \delta_2 \geq |N_{V_l \setminus S'_v}(e)|$. Therefore, $S'_v \subset V_l$ is a defensive alliance in $\mathcal{L}(\Gamma)$ and its cardinality is $\delta_1 - 1$. So, $a(\mathcal{L}(\Gamma)) \leq \delta_1 - 1$. □

Corollary 2. *If Γ is a δ-regular graph ($\delta > 0$), then $a(\mathcal{L}(\Gamma)) = \hat{a}(\mathcal{L}(\Gamma)) = \delta$.*

Theorem 3. *If Γ is a (δ_1, δ_2)-semiregular bipartite graph, then*

$$a(\mathcal{L}(\Gamma)) = \left\lceil \frac{\delta_1 + \delta_2 - 1}{2} \right\rceil \quad \text{and} \quad \hat{a}(\mathcal{L}(\Gamma)) = \left\lceil \frac{\delta_1 + \delta_2}{2} \right\rceil.$$

Proof. Suppose $\delta_1 > \delta_2$. By Theorem 1, we only need to show that there exists a defensive alliance whose cardinality is $\left\lceil \frac{\delta_1 + \delta_2 - 1}{2} \right\rceil$. Let $v \in V$ be a vertex of maximum degree in Γ and let $S_v = \{e \in E : v \in e\}$. Hence, $\langle S_v \rangle \cong K_{\delta_1}$. Therefore, taking $S \subset S_v$ such that $|S| = \left\lceil \frac{\delta_1 + \delta_2 - 1}{2} \right\rceil$, we obtain $\langle S \rangle \cong K_{\left\lceil \frac{\delta_1 + \delta_2 - 1}{2} \right\rceil}$. Thus, $\forall e \in S$,

$$|N_S(e)| + 1 = \left\lceil \frac{\delta_1 + \delta_2 - 1}{2} \right\rceil \geq \delta_1 + \delta_2 - 1 - \left\lceil \frac{\delta_1 + \delta_2 - 1}{2} \right\rceil = |N_{V_l \setminus S}(e)|.$$

So, S is a defensive alliance in $\mathcal{L}(\Gamma)$. The proof of $\hat{a}(\mathcal{L}(\Gamma)) = \left\lceil \frac{\delta_1 + \delta_2}{2} \right\rceil$ is analogous to the previous one. □

Now we are going to characterize the graphs having $a(\mathcal{L}(\Gamma)) < 4$.

Lemma 4. [3] *For any graph Γ,*

1. *$a(\Gamma) = 1$ if and only if there exists a vertex $v \in V$ such that $\delta(v) \leq 1$.*

2. *$a(\Gamma) = 2$ if and only if $2 \leq \min_{v \in V}\{\delta(v)\}$ and Γ has two adjacent vertices of degree at most three.*

3. *$a(\Gamma) = 3$ if and only if $a(\Gamma) \neq 1$, $a(\Gamma) \neq 2$, and Γ has an induced subgraph isomorphic to either (a) P_3, with vertices, in order, u, v and w, where $\delta(u)$ and $\delta(w)$ are at most three, and $\delta(v)$ is at most five, or (b) isomorphic to K_3, each vertex of which has degree at most five.*

Theorem 5. *For any graph* Γ,

1. $a(\mathcal{L}(\Gamma)) = 1$ *if and only if either* Γ *has a connected component isomorphic to* K_2*, or* Γ *has a vertex of degree one which is adjacent to a vertex of degree two.*

2. $a(\mathcal{L}(\Gamma)) = 2$ *if and only if* $a(\mathcal{L}(\Gamma)) \neq 1$ *and* Γ *has a subgraph isomorphic to* P_3*, with vertices, in order,* u, v *and* w*, such that* $\delta(u)+\delta(v) \leq 5$ *and* $\delta(v) + \delta(w) \leq 5$.

3. $a(\mathcal{L}(\Gamma)) = 3$ *if and only if* $a(\mathcal{L}(\Gamma)) \neq 1$, $a(\mathcal{L}(\Gamma)) \neq 2$*, and* Γ *has a subgraph isomorphic to either (a)* P_4*, with vertices, in order,* u, v, w *and* x*, such that* $\delta(u)+\delta(v) \leq 5$, $\delta(x)+\delta(w) \leq 5$ *and* $\delta(v)+\delta(w) \leq 7$*, or (b)* K_3*, with vertices* $\{u, v, w\}$*, such that* $\delta(u) + \delta(v) \leq 7$, $\delta(u) + \delta(w) \leq 7$ *and* $\delta(v) + \delta(w) \leq 7$*, or (c)* $K_{1,3}$*, with vertices* $\{u, v, w, x\}$*, and hub* v*, such that* $\delta(v) + \delta(u) \leq 7$, $\delta(v) + \delta(w) \leq 7$ *and* $\delta(v) + \delta(x) \leq 7$.

Proof. The result follows from Lemma 4:

1. $\mathcal{L}(\Gamma)$ has an isolated vertex if and only if Γ has a connected component isomorphic to K_2. Moreover, $\mathcal{L}(\Gamma)$ has a vertex of degree one if and only if Γ has a vertex of degree one adjacent to a vertex of degree two.

2. $\mathcal{L}(\Gamma)$ has two adjacent vertices, $e_1, e_2 \in V_l$, such that $\delta(e_1) \leq 3$ and $\delta(e_2) \leq 3$, if and only if Γ has three vertices $u, v, w \in V$ such that $e_1 = \{u, v\}$ and $e_2 = \{v, w\}$, with $\delta(u) + \delta(v) - 2 = \delta(e_1) \leq 3$ and $\delta(v) + \delta(w) - 2 = \delta(e_2) \leq 3$.

3. $\mathcal{L}(\Gamma)$ has an induced subgraph isomorphic to P_3, with vertices, in order, e_1, e_2 and e_3, where $\delta(e_1) \leq 3$, $\delta(e_2) \leq 3$ and $\delta(e_3) \leq 5$ if and only if Γ has a subgraph isomorphic to P_4, with vertices, in order, u, v, w and x, where $e_1 = \{u, v\}$, $e_2 = \{v, w\}$, $e_3 = \{w, x\}$, $\delta(u) + \delta(v) \leq 5$, $\delta(x) + \delta(w) \leq 5$ and $\delta(v) + \delta(w) \leq 7$.

   On the other hand, $\mathcal{L}(\Gamma)$ has an induced subgraph isomorphic to K_3 if and only if either Γ has a subgraph isomorphic to K_3, or Γ has a subgraph isomorphic to $K_{1,3}$. Moreover, for $e = \{u, v\} \in V_l$, $\delta(e) \leq 5$ if and only if $\delta(u) + \delta(v) \leq 7$.

□

We remark that a similar characterization can be done in the case of strong alliances.

Now we are going to compare $a(\Gamma)$ and $a(\mathcal{L}(\Gamma))$. There are cases in which $a(\Gamma) = a(\mathcal{L}(\Gamma))$. A trivial instance is the case $\Gamma \cong C_k$ (Γ isomorphic to the cycle of length k). In order to show the case $a(\mathcal{L}(\Gamma)) < a(\Gamma)$ we take $\Gamma \cong O_5$ (the odd graph O_5). That is, $a(\mathcal{L}(O_5) = 5 < 6 = girth(O_5) = a(O_5)$[1]. Moreover, there are cases in which $a(\Gamma) < a(\mathcal{L}(\Gamma))$. For instance, if either Γ is isomorphic to a tree, or Γ is isomorphic to an unicyclic[2] graph, but $\Gamma \ncong C_k$, then $1 = a(\Gamma) \leq a(\mathcal{L}(\Gamma))$. In particular, if $\Gamma \cong K_{1,n}$, $n > 2$, then $a(\Gamma) = 1 < n - 1 = a(\mathcal{L}(\Gamma))$.

Figure 1: Γ and its line graph $\mathcal{L}(\Gamma)$

We define the *characteristic set* of $S_l \subset V_l$ as $C_{S_l} := \{v \in V : v \in e, \text{ for some } e \in S_l\}$. For instance, in the graph of Figure 1, $S_l = \{f, c\}$ is a minimum defensive alliance in $\mathcal{L}(\Gamma)$ and its characteristic set, $C_{S_l} = \{1, 4, 7\}$, is a defensive alliance in Γ. Notice that C_{S_l} contains the defensive alliances $S_1 = \{1, 4\}$, $S_2 = \{1, 7\}$, $S_3 = \{4\}$ and $S_4 = \{7\}$. We emphasize that in some cases $\mathcal{L}(\Gamma)$ has not minimum defensive alliances such that its characteristic set is a defensive alliance in Γ.

Theorem 6. *If there exists a minimum defensive alliance in $\mathcal{L}(\Gamma)$ such that its characteristic set is a defensive alliance in Γ, then $a(\Gamma) \leq a(\mathcal{L}(\Gamma))$.*

Proof. Let $S_l \subset V_l$ be a minimum defensive alliance in $\mathcal{L}(\Gamma)$. We shall show that the characteristic set of S_l, C_{S_l}, contains a defensive alliance whose cardinality is $\leq |S_l|$.

As S_l is a minimum defensive alliance in $\mathcal{L}(\Gamma)$, then the subgraph $\langle S_l \rangle$ is connected and, as a consequence, the subgraph $\langle C_{S_l} \rangle$ also is connected. Therefore, $|C_{S_l}| \leq |S_l| + 1$.

Let $v \in C_{S_l}$. If $S' = C_{S_l} \backslash \{v\}$ is a defensive alliance in Γ, then $a(\Gamma) \leq a(\mathcal{L}(\Gamma))$. Suppose $S' = C_{S_l} \backslash \{v\}$ is not a defensive alliance in Γ. In such

[1] It was shown in [3] that if Γ is 5-regular, then $a(\Gamma) = girth(\Gamma)$.

[2] A connected graph containing exactly one cycle.

case, there exists $u \in S'$ such that $2|N_{S'}(u)| + 2 \leq \delta(u)$. Since $|N_{C_{S_l}}(u)| = N_{S'}(u) + 1$, we have

$$\delta(u) \geq 2|N_{C_{S_l}}(u)|. \tag{1}$$

We shall use (1) to show that $S'' = C_{S_l} \backslash \{u\}$ is a defensive alliance in Γ.

Suppose $w \in S''$ is a vertex adjacent to u and let $e = \{u, w\}$. Since S_l is a defensive alliance in $\mathcal{L}(\Gamma)$, $2|N_{S_l}(e)| + 1 \geq \delta(e)$. Therefore, by $|N_{S_l}(e)| = |N_{C_{S_l}}(u)| + |N_{C_{S_l}}(w)| - 2$ and $\delta(e) = \delta(u) + \delta(w) - 2$, we obtain

$$2|N_{C_{S_l}}(u)| + 2|N_{C_{S_l}}(w)| - 1 \geq \delta(u) + \delta(w). \tag{2}$$

By (1) and (2) we deduce $2|N_{C_{S_l}}(w)| - 1 \geq \delta(w)$. Moreover, since $|N_{C_{S_l}}(w)| = |N_{S''}(w)| + 1$, we have $2|N_{S''}(w)| + 1 \geq \delta(w)$. On the other hand, if w is not adjacent to u, then $|N_{S''}(w)| = |N_{C_{S_l}}(w)|$. Hence, $2|N_{S''}(w)| + 1 \geq \delta(w)$. Thus, S'' is a defensive alliance in Γ. $\square$

It is easy to deduce sufficient conditions for $a(\Gamma) \leq a(\mathcal{L}(\Gamma))$ or $a(\mathcal{L}(\Gamma)) \leq a(\Gamma)$ from the above bounds and the bounds on $a(\Gamma)$ obtained in [3, 4]. For instance, it was shown in [3] that $a(\Gamma) \leq \lceil \frac{n}{2} \rceil$. Hence, by Theorem 1, we have

$$\left\lceil \frac{n}{2} \right\rceil \leq \left\lceil \frac{\delta_n + \delta_{n-1} - 1}{2} \right\rceil \Rightarrow a(\Gamma) \leq a(\mathcal{L}(\Gamma)).$$

In particular,

$$\frac{n}{2} < \delta_n \Rightarrow a(\Gamma) \leq a(\mathcal{L}(\Gamma)).$$

3 Global defensive alliances and line graphs

Theorem 7. *Let Γ be a simple graph of size $m > 6$, then*

$$\gamma_a(\mathcal{L}(\Gamma)) \geq \left\lceil \sqrt{m+4} - 1 \right\rceil.$$

Proof. If S_l is a global defensive alliance in $\mathcal{L}(\Gamma)$, then

$$m - |S_l| \leq \sum_{v \in S_l} |N_{V_l \backslash S_l}(v)| \leq \sum_{v \in S_l} |N_{S_l}(v)| + |S_l| \leq |S_l|^2.$$

On the other hand, if $|S_l| \leq 2$, then $|N_{V_l \backslash S_l}(v)| \leq 2$, $\forall v \in S_l$. Thus, $m \leq 6$. Therefore, $m > 6 \Rightarrow |S_l| > 2$. By adding $3 \leq |S_l|$ and $m - |S_l| \leq |S_l|^2$, the result follows. $\square$

The above bound is attained, for instance, in the case of the graph of Figure 1. In this case we can take the minimum global defensive alliance as $S_l = \{a, b, g\}$.

Several tight bounds on $\gamma_a(\mathcal{L}(\Gamma))$ and $\gamma_{\hat{a}}(\mathcal{L}(\Gamma))$, in terms of parameters of Γ, can be derived from the previous bounds on $\gamma_a(\Gamma)$ and $\gamma_{\hat{a}}(\Gamma)$ [2, 4, 5]. For instance, we consider the following result.

Theorem 8. [4] *Let Γ be a simple graph of order n and maximum degree δ_1. Then*

$$\gamma_a(\Gamma) \geq \left\lceil \frac{2n}{\delta_1 + 3} \right\rceil \quad \textit{and} \quad \gamma_{\hat{a}}(\Gamma) \geq \left\lceil \frac{n}{\lfloor \frac{\delta_1}{2} \rfloor + 1} \right\rceil.$$

Both bounds are tight.

Corollary 9. *Let Γ be a simple graph of size m whose maximum degrees are δ_1 and δ_2. Then*

$$\gamma_a(\mathcal{L}(\Gamma)) \geq \left\lceil \frac{2m}{\delta_1 + \delta_2 + 1} \right\rceil \quad \textit{and} \quad \gamma_{\hat{a}}(\mathcal{L}(\Gamma)) \geq \left\lceil \frac{2m}{\delta_1 + \delta_2} \right\rceil.$$

In the case of connected alliances we obtain the following results.

Theorem 10. *Let $\Gamma = (V, E)$ be a connected graph of order n, size m and diameter $D(\Gamma)$. Then*

$$\gamma_{ca}(\Gamma) \geq \left\lceil \sqrt{D(\Gamma) + n} - 1 \right\rceil \textit{ and } \gamma_{ca}(\mathcal{L}(\Gamma)) \geq \left\lceil \sqrt{D(\Gamma) + m - 1} - 1 \right\rceil.$$

Proof. If S denotes a global defensive alliance in Γ, then

$$n - |S| \leq \sum_{v \in S} |N_{V \setminus S}(v)| \leq \sum_{v \in S} |N_S(v)| + |S|.$$

On the other hand, if S is a dominating set and $\langle S \rangle$ is connected, then $D(\Gamma) \leq D(\langle S \rangle) + 2$. Hence,

$$D(\Gamma) \leq |S| + 1. \tag{3}$$

By adding $n - |S| \leq |S|^2$ and (3) we obtain the bound on $\gamma_{ca}(\Gamma)$. The bound on $\gamma_{ca}(\mathcal{L}(\Gamma))$ follows from the bound on $\gamma_{ca}(\Gamma)$ and $D(\Gamma) - 1 \leq D(\mathcal{L}(\Gamma))$. $\square$

Let Γ be the left hand side graph of Figure 1. The set $S = \{1, 2, 3\}$ is a global defensive alliance in Γ and $\langle S \rangle$ is connected. On the other hand, $S_l = \{a, b, g\}$ is a global defensive alliance in $\mathcal{L}(\Gamma)$ and $\langle S_l \rangle$ is connected. In this case, Theorem 10 leads to $\gamma_{ca}(\Gamma) \geq 3$ and $\gamma_{ca}(\mathcal{L}(\Gamma)) \geq 3$. Thus, the bounds are tight.

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